1952-06

The temperature distribution in an internal combustion engine piston

Palacios, Daniel Hax

Monterey, California. Naval Postgraduate School

THE TEMPERATURE DISTRIBUTION IN AN INTERNAL COMBUSTION ENGINE PISTON

D. H. PALACIOS

THE TEMPERATURE DISTRIBUTION IN AN INTERNAL COMBUSTION ENGINE PISTON

-

D. H. PALACIOS

THE TEMPERATURE DISTRIBUTION IN AN INTERNAL COMBUSTION ENGINE PISTON

by

Daniel Hax Palacios
Lieutenant, Chilean Navy

Submitted in partial fulfillment
of the requirements
for the degree of
MASTER OF SCIENCE
in
MECHANICAL ENGINEERING

United States Naval Postgraduate School
Monterey, California
1952

This work is accepted as fulfilling

the thesis requirements for the degree of

MASTER OF SCIENCE

in

MECHANICAL ENGINEERING

from the

United States Naval Postgraduate School

P. J. Kiefer
Chairman
Department of Mechanical Engineering

Approved:

R. S. Glasgow
Academic Dean

PREFACE

This investigation was prepared during the period February - June 1952 at the United States Naval Postgraduate School, Monterey, California.

The subject was suggested to the author by previous experience in maintenance of small high-speed diesel engines, during which burnt pistons had to be replaced on several occasions.

The author wishes to extend his appreciation to Assistant Professor E. E. Drucker for his interest and advice throughout the development of the work.

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

TABLE OF SYMBOLS

Symbol	Quantity	Units
α	Thermal Diffusivity.	$ft^2\ hr^{-1}$
θ	Temperature of the Disk.	F
θ_o	Initial Temperature of Disk.	F
ϕ	Temperature of Barrel.	F
ϕ_o	Initial Temperature of Barrel.	F
ρ	Mass Density of Piston Material.	$lb_m\ ft^{-3}$
τ	Time	hr
b	Thickness of Barrel Wall.	ft
c_p	Specific Heat at Constant Pressure of Piston Material	$B\ lb_m^{-1}\ F^{-1}$
F	Degrees Fahrenheit	F
h_1	Heat-transfer Coefficient from Gas to Upper Surface of Disk	$B\ hr^{-1}\ ft^{-2}\ F^{-1}$
h_3	Heat-transfer Coefficient from outside surface of barrel into media beyond the surface	$B\ hr^{-1}\ ft^{-2}\ F^{-1}$
i	Square Root of Minus One	
J_o	Bessel function of the First Kind of Order Zero	
J_1	Bessel function of the First Kind of Order One.	
k	Thermal Conductivity of Piston Material	$B\ hr^{-1}\ ft^{-1}\ F^{-1}$
l	Thickness of Disk	ft

TABLE OF SYMBOLS

Symbol	Quantity	Units
α	Thermal Diffusivity.	$ft^2\ hr^{-1}$
θ	Temperature of the Disk.	F
θ_0	Initial Temperature of Disk.	F
ϕ	Temperature of Barrel.	F
ϕ_0	Initial Temperature of Barrel.	F
ρ	Mass Density of Piston Material.	$lb_m\ ft^{-3}$
τ	Time	hr
b	Thickness of Barrel Wall.	ft
c_p	Specific Heat at Constant Pressure of Piston Material	$B\ lb_m^{-1}\ F^{-1}$
F	Degrees Fahrenheit	F
h_1	Heat-transfer Coefficient from Gas to Upper Surface of Disk	$B\ hr^{-1}\ ft^{-2}\ F^{-1}$
h_3	Heat-transfer Coefficient from outside surface of barrel into media beyond the surface	$B\ hr^{-1}\ ft^{-2}\ F^{-1}$
i	Square Root of Minus One	
J_0	Bessel function of the First Kind of Order Zero	
J_1	Bessel function of the First Kind of Order One.	
k	Thermal Conductivity of Piston Material	$B\ hr^{-1}\ ft^{-1}\ F^{-1}$
l	Thickness of Disk	ft

L_o	Length of Barrel.	ft
n	Frequency of Gas Temperature Fluctuation	hr^{-1}
r	Radius	ft
R_o	Outside Radius of Disk	ft
Ri	Inside radius of Disk	ft
t	Temperature, Standard Scale	F
t_o	Initial temperature of Piston	F
T_a	Temperature of Cooling Medium Surrounding Barrel.	F
T_{gas}	Gas Temperature	F
T_m	Mean Temperature of Gases of Combustion.	F
T_o	Amplitude of Gas Temperature Fluctuation.	F
Y_o	Bessel Function of the second Kind of Order Zero.	
Y_1	Bessel Function of the second Kind of Order One.	
z	Length Measured from Tom of Disk Downwards.	ft

L_o	Length of Barrel.	ft
n	Frequency of Gas Temperature Fluctuation	hr^{-1}
r	Radius	ft
R_o	Outside Radius of Disk	ft
R_i	Inside radius of Disk	ft
t	Temperature, Standard Scale	F
t_o	Initial temperature of Piston	F
T_a	Temperature of Cooling Medium Surrounding Barrel.	F
T_{gas}	Gas Temperature	F
T_m	Mean Temperature of Gases of Combustion.	F
T_o	Amplitude of Gas Temperature Fluctuation.	F
Y_o	Bessel Function of the second Kind of Order Zero.	
Y_1	Bessel Function of the second Kind of Order One.	
z	Length Measured from Top of Disk Downwards.	ft

SUMMARY

The objective of this work was to establish by annalytical means the temperature distribution in an internal combustion engine piston. For this purpose the piston was divided into a disk and a barrel, for which boundary conditions were expressed for the exterior surfaces and a common conical surface. The problem was solved in two parts: first, a solution was found for the case where the gas temperature was assumed representable by an average temperature; this solution was then modified to account for a sinusoidal fluctuation of gas temperature.

The resulting temperature distribution, within the limits of the approximations made, is:

for the disk:

$$t = T_m + T_0 e^{-z\sqrt{\frac{\pi n}{\alpha}}} \sin\left(2\pi n\tau - z\sqrt{\frac{\pi n}{\alpha}}\right) +$$

$$+ \sum_{m=1}^{\infty}\sum_{p=1}^{\infty} \theta_0 e^{-\alpha(\gamma_p^2 + b_m^2)\tau}\left[\cos b_m z + \frac{h_1}{k\, b_m}\sin b_m z\right] J_0(\gamma_p r)$$

for the barrel:

$$t = T_a + T_0\, e^{-z\sqrt{\frac{\pi n}{\alpha}}} \sin\left(2\pi n\tau - z\sqrt{\frac{\pi n}{\alpha}}\right) +$$

$$+ \sum_{m=1}^{\infty}\sum_{p=1}^{\infty} P_p\, e^{-\alpha(\gamma_p^2 + b_m^2)\tau} \cos b_m(z - L_0)\left[J_0(\gamma_p r) + F_p\, Y_0(\gamma_p r)\right]$$

SUMMARY

The objective of this work was to establish by analytical means the temperature distribution in an internal combustion engine piston. For this purpose the piston was divided into a disk and a barrel, for which boundary conditions were expressed for the exterior surfaces and a common conical surface. The problem was solved in two parts: first, a solution was found for the case where the gas temperature was assumed representable by an average temperature; this solution was then modified to account for a sinusoidal fluctuation of gas temperature.

The resulting temperature distribution, within the limits of the approximations made, is:

for the disk:

$$t = T_m + T_0\, e^{-\zeta\sqrt{\frac{\pi n}{\alpha}}} \sin\left(2\pi n \tau - \zeta\sqrt{\frac{\pi n}{\alpha}}\right) +$$

$$+ \sum_{m=1}^{\infty}\sum_{p=1}^{\infty} \theta_0\, e^{-\alpha(\beta_p^2 + b_m^2)\tau}\left[\cos b_m \zeta + \frac{h_1}{k\, b_m} \sin b_m \zeta\right] J_0(\beta_p r)$$

for the barrel:

$$t = T_a + T_0\, e^{-\zeta\sqrt{\frac{\pi n}{\alpha}}} \sin\left(2\pi n \tau - \zeta\sqrt{\frac{\pi n}{\alpha}}\right) +$$

$$+ \sum_{m=1}^{\infty}\sum_{p=1}^{\infty} P_p\, e^{-\alpha(\beta_p^2 + b_m^2)\tau} \cos b_m(\zeta - l_1)\left[J_0(\beta_p r) + F_p Y_0(\beta_p r)\right]$$

CHAPTER I

INTRODUCTION

Upon analyzing a piston of an internal combustion engine with a viewpoint to understanding its behaviour as a heat dissipating member of the engine, it is readily realized that the piston absorbs a certain amount of heat through the upper surface of its crown. This heat is partly stored in the body of the piston until it reaches its steady state temperature; at the same time there is heat dissipation from the outside surface of the barrel and rings and also through its inside surfaces. After the steady state is reached, all heat absorbed must be dissipated through these surfaces.

According to O. L. Adams (1), five percent of the heat liberated by the combustion of the fuel must pass through the piston crown surface into the piston. Of this heat, ten percent is dissipated by the lower surface of the crown of the piston, as stated by J. L. Hepworth (5).

Radiation from the gases into the piston only takes place through a very small portion of the time cycle and it amounts, according to B. Pinkel (9), to only ten percent of the total heat being absorbed by all the metalic parts forming the container for the hot gases; thus it may be seen that the heat absorbed by the piston itself through radiation will be a part of this ten percent depending on the relative areas of the piston crown with respect to the area of the rest of the combustion chamber. Throughout this work it will be assumed that all heat absorption by the

CHAPTER I

INTRODUCTION

Upon analyzing a piston of an internal combustion engine with a viewpoint to understanding its behaviour as a heat dissipating member of the engine, it is readily realized that the piston absorbs a certain amount of heat through the upper surface of its crown. This heat is partly stored in the body of the piston until it reaches its steady state temperature; at the same time there is heat dissipation from the outside surface of the barrel and rings and also through its inside surfaces. After the steady state is reached, all heat absorbed must be dissipated through these surfaces.

According to O. L. Adams (1), five percent of the heat liberated by the combustion of the fuel must pass through the piston crown surface into the piston. Of this heat, ten percent is dissipated by the lower surface of the crown of the piston, as stated by J. L. Heyworth (5).

Radiation from the gases into the piston only takes place through a very small portion of the time cycle and it amounts, according to E. Finkel (9), to only ten percent of the total heat being absorbed by all the metallic parts forming the container for the hot gases; thus it may be seen that the heat absorbed by the piston itself through radiation will be a part of this ten percent depending on the relative areas of the piston crown with respect to the area of the rest of the combustion chamber. Throughout this work it will be assumed that all heat absorption by the

piston takes place by convection. Also all heat dissipated by the piston will be assumed to do so by a process of convection.

The temperature of the gases throughout the cycle varies widely; the temperature curves for an engine may be obtained either experimentally by direct measurement or annalytically from knowledge of the cycle under which the engine is operating. In this work the temperature of the gases will be assumed, for the sake of simplicity, to follow a sinusoidal variation about an average temperature.

The main objective of this work will be to determine the temperature distribution throughout the piston. For this purpose a set of boundary conditions will be prescribed which resemble as closely as possible the actual conditions under which the piston operates.

piston takes place by convection. Also all heat dissipated by the piston will be assumed to do so by a process of convection.

The temperature of the gases throughout the cycle varies widely; the temperature curves for an engine may be obtained either experimentally by direct measurement or annalytically from knowledge of the cycle under which the engine is operating. In this work the temperature of the gases will be assumed, for the sake of simplicity, to follow a sinusoidal variation about an average temperature.

The main objective of this work will be to determine the temperature distribution throughout the piston. For this purpose a set of boundary conditions will be prescribed which resemble as closely as possible the actual conditions under which the piston operates.

CHAPTER II

STATEMENT OF THE PROBLEM

1. Description of piston.

A piston with the characteristics shown in figure 1 was selected for analysis. The crown will be a flat disk of uniform thickness l. The barrel will be a cylindrical sleeve of constant thickness b.

The boundary surface of disk and barrel will be surface a-a as shown in figure 1. This surface will have properties common to both disk and barrel and this will enable some of the unknown quantities entering the questions to be found.

2. Temperature scales.

Figure 2 shows a sketch of the temperature pattern that a particular point of the disk is expected to follow. This figure is given mainly to show graphically some of the values being used throughout the development.

Figure 3 serves the same purpose as figure 2 with relation to the barrel.

The temperature of the gases will be assumed to be:

$$T_{gas} = T_m + T_o \sin 2\pi n \tau$$

where n will be the frequency of the cycle.

The temperature T_a surrounding the outside surface of the barrel will be assumed constant.

The temperatures for the disk will be measured from the level designated as   in figure 2 and will be expressed

CHAPTER II

STATEMENT OF THE PROBLEM

1. Description of piston.

A piston with the characteristics shown in figure 1 was selected for analysis. The crown will be a flat disk of uniform thickness l. The barrel will be a cylinderical sleeve of constant thickness b.

The boundary surface of disk and barrel will be surface a-a as shown in figure 1. This surface will have properties common to both disk and barrel and this will enable some of the unknown quantities entering the questions to be found.

2. Temperature scales.

Figure 2 shows a sketch of the temperature pattern that a particular point of the disk is expected to follow. This figure is given mainly to show graphically some of the values being used throughout the development.

Figure 3 serves the same purpose as figure 2 with relation to the barrel.

The temperature of the gases will be assumed to be:

$$T_{gas} = T_m + T_o \sin 2\pi n t$$

where n will be the frequency of the cycle.

The temperature T_a surrounding the outside surface of the barrel will be assumed constant.

The temperatures for the disk will be measured from the level designated as in figure 2 and will be expressed

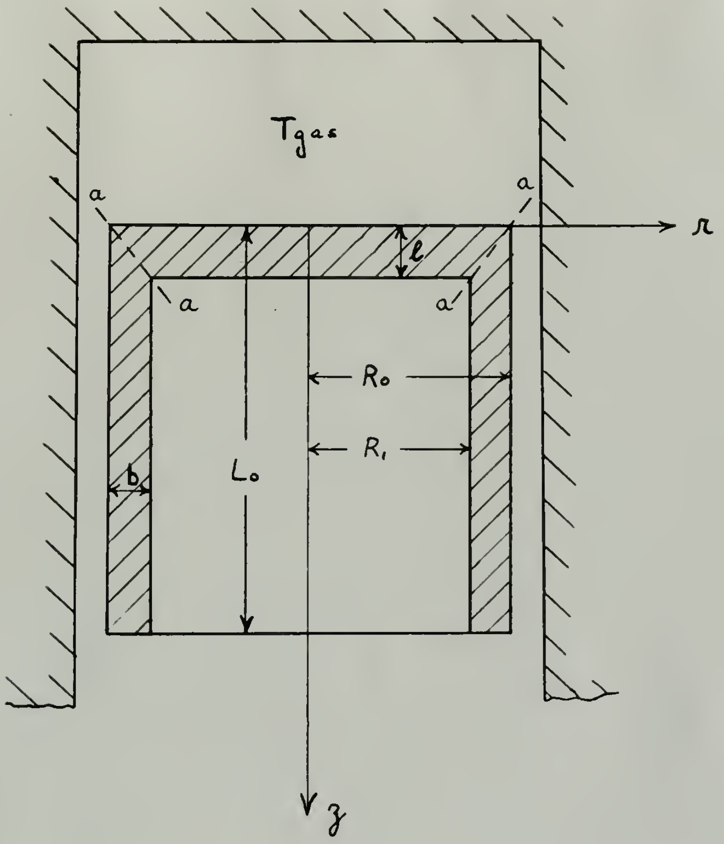

Figure 1

for any point of the disk as:

$$\theta = t - T_m$$

The initial temperature will necessarily be a constant and will be designated as θ_o.

The temperatures for points of the barrel will be measured in the ϕ scale of temperatures as shown in figure 3, where:

$$\phi = t - T_a$$

The initial temperature ϕ_o will be a constant.

It is realized that the value of T_a will initially be equal to t_o, but it is impossible with any degree of ease to consider into the problem its initial rise from t_o to T_a and thus it will be assumed constant and equal to T_a.

After a solution is reached, the results will be reverted to the standard level of measurement of temperatures in degrees F. which is designated as the t scale of temperatures.

3. Boundary conditions.

The following set of boundary conditions has been selected:

(a). At the lower surface of the disk, the rate of heat rejection is ten percent of the rate of heat absorption at the upper surface. Since the rate of heat conduction is a function of the temperature gradient normal to the surface considered, and if we assume the coefficient of heat conduction to be constant throughout the piston, we may state:

for any point of the disk as:

$$\theta = t - T_m$$

The initial temperature will necessarily be a constant and will be designated as θ_o.

The temperatures for points of the barrel will be measured in the ϕ scale of temperatures as shown in figure 3, where:

$$\phi = t - T_a$$

The initial temperature ϕ_o will be a constant.

It is realized that the value of T_a will initially be equal to t_o, but it is impossible with any degree of ease to consider into the problem its initial rise from t_o to T_a and thus it will be assumed constant and equal to T_a.

After a solution is reached, the results will be reverted to the standard level of measurement of temperatures in degrees F. which is designated as the t scale of temperatures.

3. Boundary conditions.

The following set of boundary conditions has been selected:

(a). At the lower surface of the disk, the rate of heat rejection is ten percent of the rate of heat absorption at the upper surface. Since the rate of heat conduction is a function of the temperature gradient normal to the surface considered, and if we assume the coefficient of heat conduction to be constant throughout the piston, we may state:

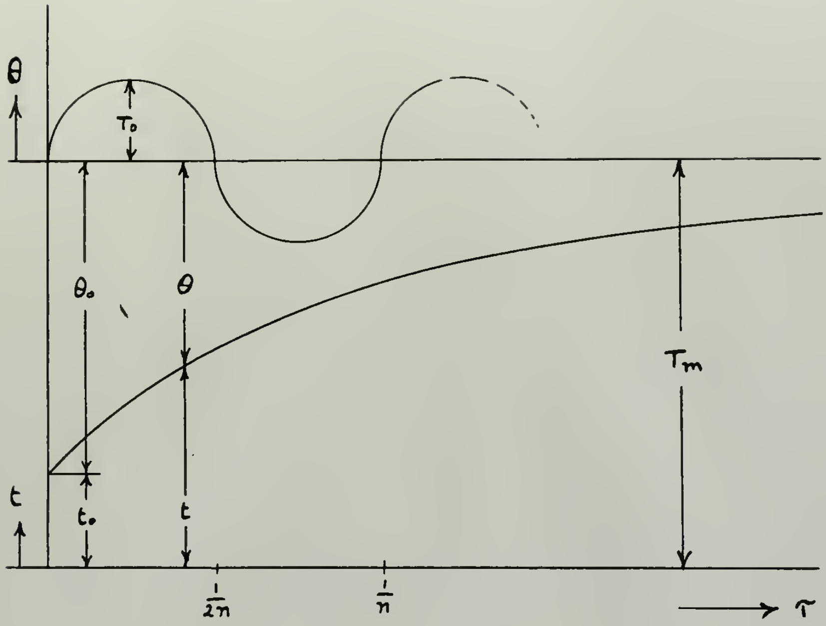

Figure 2

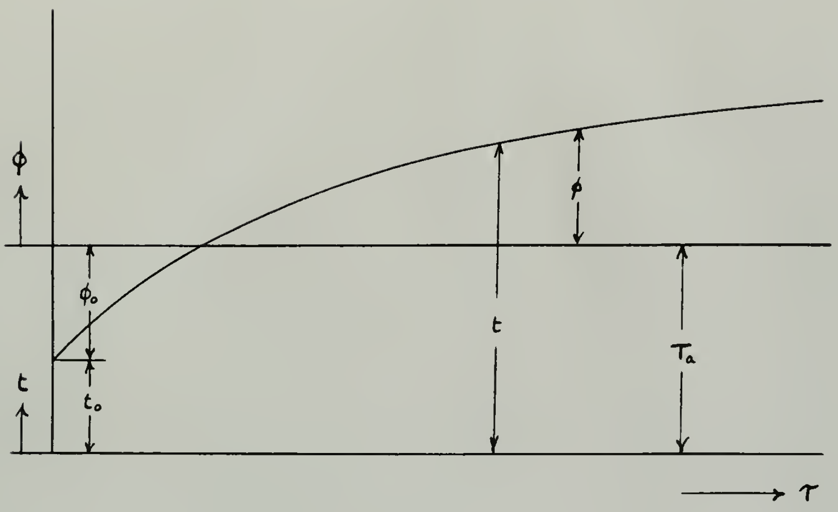

Figure 3

8

$$\left(\frac{\partial\theta}{\partial z}\right)_{z=0} = 10\left(\frac{\partial\theta}{\partial z}\right)_{z=l}$$

There will be a difference in the areas of the upper and lower surfaces determined by the relative values of R_o and R_i. In this work we will neglect this difference, but it is recognized that it could be included by multiplying the value given as 10 by the ratio of R_i to R_o squared.

(b). At the upper surface of the disk the rate of heat absorption from the gases will equal the rate of heat conduction from the surface into the disk, or:

$$k\left(\frac{\partial\theta}{\partial z}\right)_{z=0} = -h_1\left(\theta_{gas} - \theta_{z=0}\right)$$

(c). At the surface a-a the temperature of both the disk and the barrel will be equal. Since these temperatures will be measured from different levels, this relation may not be used directly without introducing undesirable constants.

Therefore it is necessary to realize that at this boundary the isotherms from disk and barrel must coincide and furthermore they must be continuous functions of the variables involved. This may be stated symbolically: along surface a-a:

$$\frac{\partial^n\theta}{\partial z^n} = \frac{\partial^n\phi}{\partial z^n}$$

and:

$$\frac{\partial^n\theta}{\partial r^n} = \frac{\partial^n\phi}{\partial r^n}$$

$$\left(\frac{d\theta}{dz}\right)_{z=0} = 10\left(\frac{d\theta}{dz}\right)_{z=c}$$

There will be a difference in the areas of the upper and lower surfaces determined by the relative values of R_o and Ri. In this work we will neglect this difference, but it is recognized that it could be included by multiplying the value given as 10 by the ratio of Ri to R_o squared.

(b). At the upper surface of the disk the rate of heat absorption from the gases will equal the rate of heat conduction from the surface into the disk, or:

$$k\left(\frac{d\theta}{dz}\right)_{z=0} = -h_1\left(\theta_{gas} - \theta_{z=0}\right)$$

(c). At the surface a-a the temperature of both the disk and the barrel will be equal. Since these temperatures will be measured from different levels, this relation may not be used directly without introducing undesirable constants.

Therefore it is necessary to realize that at this boundary the isotherms from disk and barrel must coincide and furthermore they must be continuous functions of the variables involved. This may be stated symbolically: along surface a-a:

$$\frac{d^2\theta}{dz^2} = \frac{d^2\phi}{dz^2}$$

and:
$$\frac{d^2\theta}{dr^2} = \frac{d^2\phi}{dr^2}$$

(d). The lower surface of the barrel will be assumed insulated, or:

$$\left(\frac{\partial \phi}{\partial z}\right)_{z = L_0} = 0$$

(e). The inner surface of the barrel will be assumed insulated, or:

$$\left(\frac{\partial \phi}{\partial r}\right)_{r = R_1} = 0$$

(f). At the outer surface of the barrel the rate of heat rejection from the barrel into the surrounding medium will be equal to the rate of heat conduction from the barrel into its outer surface, or:

$$k\left(\frac{\partial \phi}{\partial r}\right)_{r = R_0} = -h_3\left(\phi_{r = R_0} - \phi_a\right)$$

(g). At time = zero, a point: z = 0, r = 0, will be at a temperature θ_0, or:

at

$$\tau = 0 \qquad r = 0 \qquad z = 0 \qquad \theta = \theta_0$$

(d). The lower surface of the barrel will be assumed insulated, or:

$$\left(\frac{\partial \phi}{\partial z}\right)_{z = L_0} = 0$$

(e). The inner surface of the barrel will be assumed insulated, or:

$$\left(\frac{\partial \phi}{\partial r}\right)_{r = R_1} = 0$$

(f). At the outer surface of the barrel the rate of heat rejection from the barrel into the surrounding medium will be equal to the rate of heat conduction from the barrel into its outer surface, or:

$$k\left(\frac{\partial \phi}{\partial r}\right)_{r = R_0} = -h_3\left(\phi_{r = R_0} - \phi_a\right)$$

(g). At time = zero, a point: $z = 0$, $r = 0$, will be at a temperature θ_0, or:

at

$$T = 0 \qquad r = 0 \qquad z = 0 \qquad \theta = \theta_0$$

CHAPTER III

SOLUTION OF PROBLEM

1. General solution.

In solving the problem the disk and barrel will be treated separately and the solutions matched at the surface a-a.

Both the disk and the barrel involve heat transfer in a cylinder; the disk may be considered as a flat solid cylinder, while the barrel may be considered as a hollow cylinder. Under these conditions the best solution of Fourier's Law of Conduction of Heat will be that expressed in cylindrical coordinates:

$$\frac{\partial t}{\partial \tau} = \alpha \left(\frac{\partial^2 t}{\partial r^2} + \frac{1}{r}\frac{\partial t}{\partial r} + \frac{\partial^2 t}{\partial z^2} \right) \tag{1}$$

Assuming the variables to be separable a solution as found in Appendix I is:

$$t = E\, e^{-\alpha(a^2+b^2)\tau} \left[C \cos bz + D \sin bz \right] \left[A J_0(ar) + B Y_0(ar) \right] \tag{2}$$

From figure 2 it may be seen that the expected curves for temperatures will be of the form:

$$\theta = \theta_0\, e^{-a\tau}.$$

for a particular point on the disk.

At the same time a sinusoidal variation will be applied, which proceeding along the z-axis will be damped and out of phase; therefore our solution would have a form such as:

CHAPTER III

SOLUTION OF PROBLEM

1. General solution.

In solving the problem the disk and barrel will be treated separately and the solutions matched at the surface a-a.

Both the disk and the barrel involve heat transfer in a cylinder; the disk may be considered as a flat solid cylinder, while the barrel may be considered as a hollow cylinder. Under these conditions the best solution of Fourier's Law of Conduction of Heat will be that expressed in cylindrical coordinates:

$$\frac{\partial t}{\partial T} = \alpha\left(\frac{\partial^2 t}{\partial r^2} + \frac{1}{r}\frac{\partial t}{\partial r} + \frac{\partial^2 t}{\partial z^2}\right) \qquad (1)$$

Assuming the variables to be separable a solution as found in Appendix I is:

$$t = E\,e^{-\alpha(a^2+b^2)T}\left[C\cos bz + D\sin bz\right]\left[A J_0(ar) + B Y_0(ar)\right] \qquad (2)$$

From figure 2 it may be seen that the expected curves for temperatures will be of the form:

$$\theta = \theta_0\, e^{-aT}.$$

for a particular point on the disk.

At the same time a sinusoidal variation will be applied, which proceeding along the z-axis will be damped and out of phase; therefore our solution would have a form such as:

$$\theta = \theta_0 e^{-a\tau} + T_0 \sin(b\tau - cz) \times e^{-dz}$$

where cz is the phase angle and e^{-dz} accounts for damping.

Examining the previous equation it may be seen that the variables z and τ may not be separated as assumed in the solution found by equation (2), and thus this equation will not yield an exact solution.

A direct solution of the problem was attempted by the use of Duhamel's Theorem (3), but it was found impossible to match the result obtained for the disk with that for the barrel at the surface a-a.

Under these conditions it was decided to divide the problem into one where the gas temperature was assumed to be the average temperature T_m, and then proceed to superimpose on it the effect of the sinusoidal variation in gas temperature.

2. Solution for the disk.

From equation (2):

$$\theta = E e^{-\alpha(a^2+b^2)\tau} \left[C \cos bz + D \sin bz\right]\left[A J_0(ar) + B Y_0(ar)\right]$$

Since θ must be finite throughout the body:

$$B = 0$$

and therefore:

$$\theta = F e^{-\alpha(a^2+b^2)\tau} \left[C \cos bz + D \sin bz\right] J_0(ar) \qquad (3)$$

From boundary condition (a):

$$\left(\frac{\partial\theta}{\partial z}\right)_{z=0} = 10\left(\frac{\partial\theta}{\partial z}\right)_{z=l}$$

$$\theta = \theta_0 e^{-a\tau} + T_0 \sin(b\tau - cz) \times e^{-dz}$$

where cz is the phase angle and e^{-dz} accounts for damping.

Examining the previous equation it may be seen that the variables z and τ may not be separated as assumed in the solution found by equation (2), and thus this equation will not yield an exact solution.

A direct solution of the problem was attempted by the use of Duhamel's Theorem (3), but it was found impossible to match the result obtained for the disk with that for the barrel at the surface a-a.

Under these conditions it was decided to divide the problem into one where the gas temperature was assumed to be the average temperature T_m, and then proceed to superimpose on it the effect of the sinusoidal variation in gas temperature.

2. Solution for the disk.

From equation (2):

$$\theta = E e^{-\alpha(a^2+b^2)\tau}\left[C \cos bz + D \sin bz\right]\left[A J_0(ar) + B Y_0(ar)\right]$$

Since θ must be finite throughout the body:

$$B = 0$$

And therefore:

$$\theta = F e^{-\alpha(a^2+b^2)\tau}\left[C \cos bz + D \sin bz\right] J_0(ar) \qquad (3)$$

From boundary condition (a):

$$\left(\frac{\partial\theta}{\partial z}\right)_{z=0} = 10\left(\frac{\partial\theta}{\partial z}\right)_{z=l}$$

From equation (3):

$$\frac{\partial\theta}{\partial z} = F e^{-\alpha(a^2+b^2)\tau}\left[-Cb\sin bz + Db\cos bz\right] J_0(ar)$$

therefore:

$$\frac{D}{C} = \frac{\sin bl}{\cos bl - 0.1}$$

then:

$$\theta = G e^{-\alpha(a^2+b^2)\tau}\left[\cos bz + \frac{\sin bl}{\cos bl - 0.1}\sin bz\right] J_0(ar) \tag{4}$$

From boundary condition (b):

$$k\left(\frac{\partial\theta}{\partial z}\right)_{z=0} = -h_1\left(\theta_{gas} - \theta_{z=0}\right)$$

or:

$$\frac{k}{h_1}\left(\frac{\partial\theta}{\partial z}\right)_{z=0} = \theta_{z=0}$$

then:

$$\frac{\sin bl}{\cos bl - 0.1} = \frac{h_1}{kb}$$

The solution of this equation for b may only be done graphically; it will yield a series of values for b which will be designated as b_m.

Equation (4) now becomes:

$$\theta = \sum_{m=1}^{\infty} G e^{-\alpha(a^2+b^2)\tau}\left[\cos b_m z + \frac{h_1}{kb_m}\sin b_m z\right] J_0(ar) \tag{5}$$

From boundary condition (g):

$$\tau = 0 \qquad z = 0 \qquad r = 0 \qquad \theta = \theta_0$$

Substituting this in equation (5):

$$G = \theta_0$$

From equation (3):

$$\frac{d\theta}{dy} = F e^{-\alpha(a^2+b^2)T}\left[-Cb \sin by + Db \cos by\right] J_0(ax)$$

therefore:

$$\frac{D}{C} = \frac{\sin bl}{\cos bl - 0.1}$$

then:

$$\theta = G e^{-\alpha(a^2+b^2)T}\left[\cos by + \frac{\sin bl}{\cos bl - 0.1} \sin by\right] J_0(ax) \tag{4}$$

From boundary condition (b):

$$k\left(\frac{d\theta}{dy}\right)_{y=0} = -h_1\left(\theta_{gas} - \theta_{y=0}\right)$$

or:

$$\frac{k}{h_1}\left(\frac{d\theta}{dy}\right)_{y=0} = \theta_{y=0}$$

then:

$$\frac{\sin bl}{\cos bl - 0.1} = \frac{h_1}{kb}$$

The solution of this equation for b may only be done graphically; it will yield a series of values for b which will be designated as b_m.

Equation (4) now becomes:

$$\theta = \sum_{m=1}^{\infty} G e^{-\alpha(a^2+b^2)T}\left[\cos b_m y + \frac{h_1}{k b_m} \sin b_m y\right] J_0(ax) \tag{5}$$

From boundary condition (c):

$T = 0 \quad y = 0 \quad x = 0 \quad \theta = \theta_0$

Substituting this in equation (5):

$$G = \theta_0$$

Therefore:

$$\theta = \sum_{m=1}^{\infty} \theta_0 \, e^{-\alpha(a^2+b_m^2)\tau} \left[\cos b_m z + \frac{h_1}{k\, b_m} \sin b_m z\right] J_0(ar) \tag{6}$$

3. Solution for the barrel.

From Equation (2):

$$\phi = M e^{-\alpha(f^2+g^2)\tau} \left[\cos g z + L \sin g z\right]\left[R J_0(fr) + N Y_0(fr)\right] \tag{7}$$

From boundary condition (d):

$$\left(\frac{\partial \phi}{\partial z}\right)_{z=L_0} = 0$$

therefore:

$$L = \tan g L_0$$

then:

$$\phi = P e^{-\alpha(f^2+g^2)\tau} \cos g(z-L_0)\left[R J_0(fr) + N Y_0(fr)\right] \tag{8}$$

From boundary condition (e):

$$\left(\frac{\partial \phi}{\partial r}\right)_{r=R_1} = 0$$

or:

$$\frac{N}{R} = -\frac{J_1(fR_1)}{Y_1(fR_1)} = F$$

therefore:

$$\phi = Q e^{-\alpha(f^2+g^2)\tau} \cos g(z-L_0)\left[J_0(fr) + F\, Y_0(fr)\right] \tag{9}$$

From boundary condition (f):

$$k\left(\frac{\partial \phi}{\partial r}\right)_{r=R_0} = -h_3\left(\phi_{r=R_0} - 0\right)$$

or: $$\frac{f k}{h_3} = \frac{J_0(f R_0) + F\, Y_0(f R_0)}{J_1(f R_0) + F\, Y_1(f R_0)}$$

Solving this equation graphically for the value of f, a set of values f_p are found that satisfy this equation, and equation (9) becomes:

$$\phi = \sum_{p=1}^{\infty} P\, e^{-\alpha(f_p^2 + g^2)\tau} \cos g(z - L_0)\left[J_0(f_p r) + F_p\, Y_0(f_p r)\right] \qquad (10)$$

4. Conditions at the surface a-a.

Recapitulating:

$$\theta = \sum_{m=1}^{\infty} \theta_0\, e^{-\alpha(a^2 + b_m^2)\tau}\left[\cos b_m z + \frac{h_1}{k b_m} \sin b_m z\right] J_0(a r) \qquad (6)$$

and:

$$\phi = \sum_{p=1}^{\infty} P\, e^{-\alpha(f_p^2 + g^2)\tau} \cos g(z - L_0)\left[J_0(f_p r) + F_p\, Y_0(f_p r)\right] \qquad (10)$$

From boundary condition (c) at points along the surface a-a:

$$\frac{\partial^n \theta}{\partial z^n} = \frac{\partial^n \phi}{\partial z^n} \qquad (11)$$

$$\frac{\partial^n \theta}{\partial r^n} = \frac{\partial^n \phi}{\partial r^n} \qquad (12)$$

For equation (11) it is easily seen that for n = 1 and for a particular point along the surface a-a:

$$\sum_{m=1}^{\infty} T_m = \sum_{p=1}^{\infty} S_p \qquad (13)$$

then for n = 5:

$$\sum_{m=1}^{\infty} b_m^4\, T_m = \sum_{p=1}^{\infty} g^4 S_p \qquad (14)$$

this suggests that as a method of approach to the evaluation of a, g, and P we assume:

$$g = b_m$$

or:

$$\frac{fk}{k_s} = \frac{J_0(fR_0) + F\,Y_0(fR_0)}{J_1(fR_0) + F\,Y_1(fR_0)}$$

Solving this equation graphically for the value of f, a set of values f_p are found that satisfy this equation, and equation (9) becomes:

$$\phi = \sum_{p=1}^{\infty} P\, e^{-\alpha(f_p^2+g^2)t} \cos g(y-L_0)\left[J_0(f_p r) + F_p\,Y_0(f_p r)\right] \qquad (10)$$

4. Conditions at the surface a-a.

Recapitulating:

$$\theta = \sum_{m=1}^{\infty} \theta_0\, e^{-\alpha(a^2+b_m^2)t}\left[\cos b_m y + \frac{h_1}{k b_m}\sin b_m y\right] J_0(ar) \qquad (6)$$

and:

$$\phi = \sum_{p=1}^{\infty} P\, e^{-\alpha(f_p^2+g^2)t} \cos g(y-L_0)\left[J_0(f_p r) + F_p\,Y_0(f_p r)\right] \qquad (10)$$

From boundary condition (c) at points along the surface a-a:

$$\frac{d^n\theta}{dy^n} = \frac{d^n\phi}{dy^n} \qquad (11)$$

$$\frac{d^n\theta}{dr^n} = \frac{d^n\phi}{dr^n} \qquad (12)$$

For equation (11) it is easily seen that for n = 1 and for a particular point along the surface a-a:

$$\sum_{m=1}^{\infty} T_m = \sum_{p=1}^{\infty} S_p \qquad (13)$$

then for n = 2:

$$\sum_{m=1}^{\infty} b_m^2 T_m = \sum_{p=1}^{\infty} g^2 S_p \qquad (14)$$

this suggests that as a method of approach to the evaluation of a, g, and P we assume:

$g = b_m$

Acceptance of this assumption involves the necessity, in order to maintain the equality of the above equation, that:

$$T_m = S_p$$

where:

$$m = p$$

Also, in order that the effect of time on the temperature be equal for the same point on the surface a-a as approached by either equation (6) or (10), that:

$$f_p = a$$

Applying this condition to equation (13), we find:

$$\theta_0\left[\sin b_m z - \frac{h_1}{k b_m}\cos b_m z\right] = P \sin b_m (z - L_0)\left[1 - \frac{J_1(f_p R_1)\,Y_0(f_p r)}{Y_1(f_p R_1)\,J_0(f_p r)}\right] \tag{15}$$

where:

$$m = p$$

This equation is valid for points along the surface a-a, where:

$$z = \frac{l}{b}(R_0 - r)$$

and should yield the values of Pp.

Now, the replacement of equation (15) into equation (14) will yield an equation in terms of trigonometric and Bessel functions of r, valid for values of r ranging from R_1 to R_o. Since this equation may not be simplified to any appreciable extent, it has been considered more advisable to replace the values of z and r of a particular point along the surface a-a, such that:

$$z = l/2 = l_a$$

$$r = (R_1 + R_o)/2 = R_a$$

Acceptance of this assumption involves the necessity, in order to maintain the equality of the above equation, that:

$$T_m = S_p$$

where: $m = p$

Also, in order that the effect of time on the temperature be equal for the same point on the surface a-a as approached by either equation (6) or (10), that:

$$f_p = a$$

Applying this condition to equation (13), we find:

$$\theta_0\left[\sin b_m\xi - \frac{h_1}{k\,b_m}\cos b_m\xi\right] = F \sin b_m(\xi - L_0)\left[1 - \frac{J_1(f_pR_1)Y_0(f_pr)}{Y_1(f_pR_1)J_0(f_pr)}\right] \qquad (15)$$

where:

$$m = p$$

This equation is valid for points along the surface a-a, where:

$$\xi = \frac{\lambda}{b}(R_0 - r)$$

and should yield the values of F_p.

Now, the replacement of equation (15) into equation (14) will yield an equation in terms of trigonometric and Bessel functions of r, valid for values of r ranging from R_1 to R_0. Since this equation may not be simplified to any appreciable extent, it has been considered more advisable to replace the values of z and r of a particular point along the surface a-a, such that:

$$z = \lambda/2 = \lambda_a$$

$$r = (R_1 + R_0)/2 = R_a$$

then we find:

$$\theta_0\left[\sin b_m l_a - \frac{h_1}{k b_m}\cos b_m l_a\right] = P_p \sin b_m (l_a - L_0)\left[1 - \frac{J_1(f_p R_1)\,Y_0(f_p R_a)}{Y_1(f_p R_1)\,J_0(f_p R_a)}\right] \qquad (16)$$

where: $m = p$

If now in equation (12) for $n = 1$ this same point in the surface a-a is investigated:

$$\theta_0\left[\cos b_m l_a - \frac{h_1}{k b_m}\sin b_m l_a\right] = P_p \cos b_m (l_a - L_0)\left[1 - \frac{J_1(f_p R_1)\,Y_1(f_p R_a)}{Y_1(f_p R_1)\,J_1(f_p R_a)}\right] \qquad (17)$$

Equations (16) and (17) should give the same value for P_p. There is no way of proving that this will be so except by working out a particular problem. If in so doing it is found that the values of P_p satisfy simultaneously both equations, this will then mean that the assumption made that: $g = b_m$

and the derived expressions:

$$T_m = S_p$$

and: $f_p = a$

are justified.

In case that the values of P_p found by means of one of the equations mentioned do not satisfy the other equation, there is still the possibility of introducing this value in the second equation and determining a relationship between R_1, b, L_o, and l such that both equations are satisfied and for which the assumption made is valid. This would introduce a limitation to the physical proportions of the pistons for which this development is useful. The relationship just mentioned would take the form:

then we find:

$$\theta_0\left[\sin b_m l_a - \frac{h_1}{k b_m}\cos b_m l_a\right] = F_p \sin b_m (l_a - l_0)\left[1 - \frac{J_1(f_p R_1)\,Y_0(f_p R_a)}{Y_1(f_p R_1)\,J_0(f_p R_a)}\right] \qquad (16)$$

where: $m = p$

If now in equation (13) for $n = 1$ this same point in the surface a-a is investigated:

$$\theta_0\left[\cos b_m l_a - \frac{h_1}{k b_m}\sin b_m l_a\right] = F_p \cos b_m (l_a - l_0)\left[1 - \frac{J_1(f_p R_1)\,Y_1(f_p R_a)}{Y_1(f_p R_1)\,J_1(f_p R_a)}\right] \qquad (17)$$

Equations (16) and (17) should give the same value for F_p. There is no way of proving that this will be so except by working out a particular problem. If in so doing it is found that the values of F_p satisfy simultaneously both equations, this will then mean that the assumption made that:

$$S = b_m$$

and the derived expressions:

$$T_m = S_p$$

and:

$$f_p = a$$

are justified.

In case that the values of F_p found by means of one of the equations mentioned do not satisfy the other equation, there is still the possibility of introducing this value in the second equation and determining a relationship between R_1, b, L_0, and l such that both equations are satisfied and for which the assumption made is valid. This would introduce a limitation to the physical proportions of the pistons for which this development is useful. The relationship just mentioned would take the form:

$$\frac{\sin b_m l_a - \frac{h_1}{k b_m}\cos b_m l_a}{\cos b_m l_a + \frac{h_1}{k b_m}\sin b_m l_a} = \tan b_m (l_a - L_o)\,\frac{1 - \frac{J_1(f_p R_1)\,Y_0(f_p R_a)}{Y_1(f_p R_1)\,J_0(f_p R_a)}}{1 - \frac{J_1(f_p R_1)\,Y_1(f_p R_a)}{Y_1(f_p R_1)\,J_1(f_p R_a)}} \tag{18}$$

5. Partial solution using T_m.

On the light of previous conclusions, the solution is now:

$$\theta = \sum_{m=1}^{\infty}\sum_{p=1}^{\infty} \theta_o\, e^{-\alpha(f_p^2 + b_m^2)\tau}\left[\cos b_m z + \frac{h_1}{k b_m}\sin b_m z\right] J_0(f_p r) \tag{19}$$

$$\phi = \sum_{m=1}^{\infty}\sum_{p=1}^{\infty} P_p\, e^{-\alpha(f_p^2 + b_m^2)\tau}\cos b_m (z - L_o)\left[J_0(f_p r) + F_p Y_0(f_p r)\right] \tag{20}$$

and since: $\theta = t - T_{gas}$

and: $\phi = t - T_a$

we find, for the disk:

$$t = T_{gas} + \sum_{m=1}^{\infty}\sum_{p=1}^{\infty} \theta_o\, e^{-\alpha(f_p^2 + b_m^2)\tau}\left[\cos b_m z + \frac{h_1}{k b_m}\sin b_m z\right] J_0(f_p r) \tag{21}$$

and for the barrel:

$$t = T_a + \sum_{m=1}^{\infty}\sum_{p=1}^{\infty} P_p\, e^{-\alpha(f_p^2 + b_m^2)\tau}\cos b_m (z - L_o)\left[J_0(f_p r) + F\, Y_0(f_p r)\right] \tag{22}$$

6. Superposition of sinusoidal variation.

In order to superimpose the effect of the sinusoidal fluctuation of the gas temperature about a mean temperature T_m, the above equation (21) in a simplified form:

$$t = T_{gas} + f(r, z, \tau)$$

$$(18)\qquad \frac{1-\dfrac{J_1(\gamma_p R_1)\,Y_0(\gamma_p R_0)}{Y_1(\gamma_p R_1)\,J_0(\gamma_p R_0)}}{1-\dfrac{J_1(\gamma_p R_1)\,Y_1(\gamma_p R_0)}{Y_1(\gamma_p R_1)\,J_1(\gamma_p R_0)}}\,\tan b_m(l_w - l_o) = \frac{\sin b_m l_o - \dfrac{h_1}{k b_m}\cos b_m l_o}{\cos b_m l_o + \dfrac{h_1}{k b_m}\sin b_m l_o}$$

6. Partial solution using T_m.

On the light of previous conclusions, the solution is now:

$$(19)\qquad \theta = \sum_{m=1}^{\infty}\sum_{p=1}^{\infty} B_0\, e^{-\alpha(\gamma_p^2 + b_m^2)\tau}\left[\cos b_m \xi + \frac{h_1}{k b_m}\sin b_m \xi\right] J_0(\gamma_p r)$$

$$(20)\qquad \phi = \sum_{m=1}^{\infty}\sum_{p=1}^{\infty} F_p\, e^{-\alpha(\gamma_p^2 + b_m^2)\tau}\cos b_m(\xi - l_o)\left[J_0(\gamma_p r) + F_p Y_0(\gamma_p r)\right]$$

and since:

$$\theta = t - T_{gas}$$

and:

$$\phi = t - T_a$$

we find, for the disc:

$$(21)\qquad t = T_{gas} + \sum_{m=1}^{\infty}\sum_{p=1}^{\infty} B_0\, e^{-\alpha(\gamma_p^2 + b_m^2)\tau}\left[\cos b_m \xi + \frac{h_1}{k b_m}\sin b_m \xi\right] J_0(\gamma_p r)$$

and for the barrel:

$$(22)\qquad t = T_a + \sum_{m=1}^{\infty}\sum_{p=1}^{\infty} F_p\, e^{-\alpha(\gamma_p^2 + b_m^2)\tau}\cos b_m(\xi - l_o)\left[J_0(\gamma_p r) + F\, Y_0(\gamma_p r)\right]$$

6. Superposition of sinusoidal variation.

In order to superimpose the effect of the sinusoidal fluctuation of the gas temperature about a mean temperature T_m, the above equation (21) in a simplified form:

$$t = T_{gas} + f(r, \xi, \tau)$$

after a quarter of a cycle, or at $\tau = \frac{1}{4n}$ we find that:

$$T_{gas} = T_m + T_o \sin 2\pi n \tau$$

therefore:

$$T_{gas} = T_m + T_o$$

If this were put into equation (21) it would give an answer corresponding to the situation where this new temperature would have been imposed on the piston since time zero.

To avoid this, instead of:

$$T_o \sin 2\pi n \tau \tag{23}$$

use the following expression given by M. Jakob and G. A. Hawkins (7):

$$T_o e^{-z\sqrt{\frac{\pi n}{\alpha}}} \sin \left(2\pi n \tau - z\sqrt{\frac{\pi n}{\alpha}}\right) \tag{24}$$

where:

T_o is the amplitude of the oscillation at the surface of the disk;

$e^{-z\sqrt{\frac{\pi n}{\alpha}}}$ accounts for damping effects along z, and

$-z\sqrt{\frac{\pi n}{\alpha}}$ accounts for the change in phase along z.

The above expression was given for a thick plate subjected to conditions similar to those of this problem. Even if in this case the disk may perhaps not be considered a thick plate, it is felt that this equation will yield a fairly good approximation if it is considered that it is common knowledge that the effects of fluctuating surface temperatures do not penetrate to any great extent beyond the surface exposed. In this particular case this effect is further minimized by the fact that the fluctuation of temperatures is very rapid.

after a quarter of a cycle, or at $\tau = \frac{1}{4n}$ we find that:

$$T_{gas} = T_m + T_o \sin 2\pi n\tau$$

therefore:

$$T_{gas} = T_m + T_o$$

If this were put into equation (21) it would give an answer corresponding to the situation where this new temperature would have been imposed on the piston since time zero.

To avoid this, instead of:

$$T_o \sin 2\pi n \tau \tag{23}$$

use the following expression given by M. Jakob and G. A. Hawkins (7):

$$T_o\, e^{-\mathfrak{z}\sqrt{\frac{\pi n}{\alpha}}} \sin\left(2\pi n\tau - \mathfrak{z}\sqrt{\frac{\pi n}{\alpha}}\right) \tag{24}$$

where:

T_o is the amplitude of the oscillation at the surface of the disk;

$e^{-\mathfrak{z}\sqrt{\frac{\pi n}{\alpha}}}$ accounts for damping effects along z, and

$-\mathfrak{z}\sqrt{\frac{\pi n}{\alpha}}$ accounts for the change in phase along z.

The above expression was given for a thick plate subjected to conditions similar to those of this problem. Even if in this case the disk may perhaps not be considered a thick plate, it is felt that this equation will yield a fairly good approximation if it is considered that it is common knowledge that the effects of fluctuating surface temperatures do not penetrate to any great extent beyond the surface exposed. In this particular case this effect is further minimized by the fact that the fluctuation of temperatures is very rapid.

From equations (23) and (24) it may be seen that what happens at the surface at time zero will have its effect at a depth z with a time lag of:

$$\Delta\tau = \tfrac{1}{2}\, z \sqrt{\frac{1}{\alpha\pi n}}$$

The addition of the term expressed in equation (24) will affect in equal form both the equations (21) and (22). Since for the barrel the values of z will, with the exception of its topmost part, be far greater than those for the disk, this effect will only be perceptible in its upper section.

Including equation (24) into equations (21) and (22) we have as a final result:

For the disk:

$$t = T_m + T_o\, e^{-z\sqrt{\frac{\pi n}{\alpha}}} \sin\left(2\pi n\tau - z\sqrt{\frac{\pi n}{\alpha}}\right) + \sum_{m=1}^{\infty}\sum_{p=1}^{\infty} \theta_o\, e^{-\alpha(\gamma_p^2+b_m^2)\tau}\left[\cos b_m z + \frac{h_1}{k b_m}\sin b_m z\right] J_o(\gamma_p r)$$

For the barrel:

$$t = T_a + T_o\, e^{-z\sqrt{\frac{\pi n}{\alpha}}} \sin\left(2\pi n\tau - z\sqrt{\frac{\pi n}{\alpha}}\right) + \sum_{m=1}^{\infty}\sum_{p=1}^{\infty} P_p\, e^{-\alpha(\gamma_p^2+b_m^2)\tau} \cos b_m (z - L_o)\left[J_o(\gamma_p r) + F_p Y_o(\gamma_p r)\right]$$

7. Conclusions.

It is realized that the results obtained are not rigorous but it is felt that they constitute a fairly good approximation to the actual temperature distribution in the piston.

Further work might be done on the subject by solving an actual problem in order to check the practicability of

From equations (23) and (24) it may be seen that what happens at the surface at time zero will have its effect at a depth z with a time lag of:

$$\Delta\tau = \tfrac{1}{2} z \sqrt{\frac{1}{\pi n \alpha}}$$

The addition of the term expressed in equation (24) will affect in equal form both the equations (21) and (22). Since for the barrel the values of z will, with the exception of its topmost part, be far greater than those for the disk, this effect will only be perceptible in its upper section.

Including equation (24) into equations (21) and (22) we have as a final result:

For the disk:

$$t = T_m + T_0 e^{-z\sqrt{\frac{\pi n}{\alpha}}} \sin\left(2\pi n\tau - z\sqrt{\frac{\pi n}{\alpha}}\right) +$$

$$+ \sum_{m=1}^{\infty}\sum_{p=1}^{\infty} B_p e^{-\alpha(\beta_p^2 + b_m^2)\tau}\left[\cos b_m z + \frac{h_1}{k b_m}\sin b_m z\right] J_0(\beta_p r)$$

For the barrel:

$$t = T_a + T_0 e^{-z\sqrt{\frac{\pi n}{\alpha}}} \sin\left(2\pi n\tau - z\sqrt{\frac{\pi n}{\alpha}}\right) +$$

$$+ \sum_{m=1}^{\infty}\sum_{p=1}^{\infty} q_p e^{-\alpha(\beta_p^2 + b_m^2)\tau}\cos b_m (z - L_0)\left[J_0(\beta_p r) + r_p Y_0(\beta_p r)\right]$$

7. Conclusions.

It is realized that the results obtained are not rigorous but it is felt that they constitute a fairly good approximation to the actual temperature distribution in the piston.

Further work might be done on the subject by solving an actual problem in order to check the practicability of

the method used to determine P_p. By actual experiment on an engine the overall results of this work could be tested for accuracy and the percentage of error, if any could be found.

BIBLIOGRAPHY

1. Adams, O.L. Elements of Diesel Engineering. New York, Henley Publishing Co., 1936.

2. Boelter, L.M.K., and others. Heat Transfer Notes. Berkeley, University of California Press, 1948.

3. Carslaw, H.S. The Conduction of Heat. London, Macmillan and Co., 1921.

4. Gray, A., G.B. Mathews, and T.M. Macrobert. A Treatise on Bessel Functions. London, Macmillan and Co., 1931.

5. Hepworth, J.L. Piston Assemblies for Road Transport Oil Engines. Engineering 168:629-632. London, December 9th., 1949.

6. Ingersoll, L.R., and O.J. Zobel. Heat Conduction. New York, McGraw Hill, 1948.

7. Jakob, M., and G.A. Hawkins, Elements of Heat Transfer and Insulation. New York, John Wiley and Sons, 1950.

8. McAdams, W.H., Heat Transmission. New York, McGraw Hill, 1942.

9. Pinkel, B. Heat Transfer Processes in Air-cooled Engine Cylinders. Washington, National Advisory Committee for Aeronautics, Report No. 612, 1938.

10. Sokolnikoff, I.S., and E.S. Higher Mathematics for Engineers and Physicists. New York, McGraw Hill, 1941.

APPENDIX I

GENERAL SOLUTION

To find a general solution to Fourier's Law of Heat Conduction for the case where the temperature should be analyzed in the unsteady state for a body best described in terms of cylindrical coordinates, we use:

$$\frac{\partial t}{\partial \tau} = \alpha \left(\frac{\partial^2 t}{\partial r^2} + \frac{1}{r} \frac{\partial t}{\partial r} + \frac{\partial^2 t}{\partial z^2} \right) \tag{1}$$

where:

$$t = f(r, z, \tau)$$

Assume:

$$t = \theta(\tau) \times R(r) \times Z(z)$$

then:

$$\frac{1}{\alpha \theta} \frac{d\theta}{d\tau} = \frac{1}{R} \frac{d^2 R}{dr^2} + \frac{1}{rR} \frac{dR}{dr} + \frac{1}{Z} \frac{d^2 Z}{dz^2} \tag{2}$$

Since r, z, and τ are independent variables, we may say:

$$\frac{1}{R} \frac{d^2 R}{dr^2} + \frac{1}{rR} \frac{dR}{dr} = -a^2 \tag{3}$$

$$\frac{1}{Z} \frac{d^2 Z}{dz^2} = -b^2 \tag{4}$$

$$\frac{1}{\alpha \theta} \frac{d\theta}{d\tau} = -a^2 - b^2 \tag{5}$$

Equation (3) may be rearranged as:

$$\frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} + a^2 R = 0$$

which is Bessel's equation of the first kind of order zero.

APPENDIX I

GENERAL SOLUTION

To find a general solution to Fourier's Law of Heat Conduction for the case where the temperature should be analyzed in the unsteady state for a body best described in terms of cylindrical coordinates, we use:

(1) $$\frac{dt}{d\tau} = \alpha\left(\frac{d^2t}{dr^2} + \frac{1}{r}\frac{dt}{dr} + \frac{d^2t}{dz^2}\right)$$

where:

$$t = f(r, z, \tau)$$

Assume: $$t = \theta(\tau) \times R(r) \times Z(z)$$

then:

(2) $$\frac{1}{\alpha\theta}\frac{d\theta}{d\tau} = \frac{1}{R}\frac{d^2R}{dr^2} + \frac{1}{rR}\frac{dR}{dr} + \frac{1}{Z}\frac{d^2Z}{dz^2}$$

Since r, z, and τ are independent variables, we may say:

(3) $$\frac{1}{R}\frac{d^2R}{dr^2} + \frac{1}{rR}\frac{dR}{dr} = -a^2$$

(4) $$\frac{1}{Z}\frac{d^2Z}{dz^2} = -b^2$$

(5) $$\frac{1}{\alpha\theta}\frac{d\theta}{d\tau} = -a^2 - b^2$$

Equation (3) may be rearranged as:

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + a^2R = 0$$

which is Bessel's equation of the first kind of order zero.

Then: $$R = A\,J_0(ar) + B\,Y_0(ar) \tag{6}$$

Equation (4) may be rearranged as:

$$\frac{d^2 Z}{dz^2} + b^2 Z = 0$$

which yields: $$Z = C \cos bz + D \sin bz \tag{7}$$

Equation (5) may be rearranged as:

$$\frac{d\theta}{d\tau} + \alpha\,\theta\,(a^2 + b^2) = 0$$

which yields: $$\theta = E\, e^{-\alpha (a^2+b^2)\tau} \tag{8}$$

But, since: $t = \theta \times R \times Z$

we find using equations (6), (7), and (8), that:

$$t = E\, e^{-\alpha(a^2+b^2)\tau}\left[C \cos bz + D \sin bz\right]\left[A\,J_0(ar) + B\,Y_0(ar)\right]$$

In this equation the constants a, b, A, B, C, D, and E must be found so as to satisfy the boundary conditions of the specific problem involved.

18054

Thesis P14

Palacios

The temperature distribution in an internal combustion engine piston.